Uniform convergence on a Bakhvalov-type mesh using the preconditioning approach: Technical report

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Abstract

The linear singularly perturbed convection-diffusion problem in one dimension is considered and its discretization on a Bakhvalov-type mesh is analyzed. The preconditioning technique is used to obtain the pointwise convergence uniform in the perturbation parameter.

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1 Introduction

The report is a supplement to [8].

2 The continuous problem

We consider the problem

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1}$$

with a small positive perturbation parameter ε and $C^1[0,1]$-functions b, c, and f, where b and c satisfy

$$b(x) \geq \beta > 0, \quad c(x) \geq 0 \ \text{ for } x \in I := [0,1].$$

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It is well known, see [3, 5] for instance, that (1) has a unique solution u in $C^3(I)$, which in general has a boundary layer near $x = 0$. Our goal is to find this solution numerically.

The solution u can be decomposed into the smooth and boundary-layer parts. We present here Linß's [4, Theorem 3.48] version of such a decomposition:

$$u(x) = s(x) + y(x), \tag{2}$$

$$|s^{(k)}(x)| \le C\left(1 + \varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \tag{3}$$
$$x \in I, \quad k = 0, 1, 2, 3.$$

Above and throughout the report, C denotes a generic positive constant which is independent of ε. For the construction of the function s, see [4], since the details are not of interest here. As for y, it is important to note that it solves the problem

$$\mathcal{L}y(x) = 0, \quad x \in (0,1), \quad y(0) = -s(0), \quad y(1) = 0,$$

with a homogeneous differential equation. We shall use this fact later on in the report.

3 The discrete problem and condition number estimate

We first define a finite-difference discretization of the problem (1) on a general mesh I^N with mesh points x_i, $i = 0, 1, \dots, N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. Throughout the rest of the paper, the constants C are also independent of N.

Let $h_i = x_i - x_{i-1}$, $i = 1, 2, \dots, N$, and $\hbar_i = (h_i + h_{i+1})/2$, $i = 1, 2, \dots, N-1$. Mesh functions on I^N are denoted by W^N, U^N, etc. If g is a function defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function. Any mesh function W^N is identified with an $(N+1)$-dimensional column vector, $W^N = [W_0^N, W_1^N, \dots, W_N^N]^T$, and its maximum norm is given by

$$\left\|W^N\right\| = \max_{0 \le i \le N} |W_i^N|.$$

For the matrix norm, which we also denote by $\|\cdot\|$, we take the norm subordinate to the above maximum vector norm.

We discretize the problem (1) on I^N using the upwind finite-difference scheme:

$$\begin{aligned} U_0^N &= 0, \\ \mathcal{L}^N U_i^N := -\varepsilon D'' U_i^N - b_i D' U_i^N + c_i U_i^N &= f_i, \quad i = 1, 2, \dots, N-1, \\ U_N^N &= 0, \end{aligned} \tag{4}$$

where

$$D'' W_i^N = \frac{1}{\hbar_i}\left(\frac{W_{i+1}^N - W_i^N}{h_{i+1}} - \frac{W_i^N - W_{i-1}^N}{h_i}\right)$$

and

$$D' W_i^N = \frac{W_{i+1}^N - W_i^N}{h_{i+1}}.$$

The linear system (4) can be written down in matrix form,

$$A_N U^N = \hat{f}^N, \tag{5}$$

where $A_N = [a_{ij}]$ is a tridiagonal matrix with $a_{00} = 1$ and $a_{NN} = 1$ being the only nonzero elements in the 0th and Nth rows, respectively, and where $\hat{f}^N = [0, f_1, f_2, \ldots, f_{N-1}, 0]^T$.

It is easy to see that A_N is an L-matrix, i.e., $a_{ii} > 0$ and $a_{ij} \leq 0$ if $i \neq j$, for all $i, j = 0, 1, \ldots, N$. The matrix A_N is also inverse monotone, which means that it is non-singular and that $A_N^{-1} \geq 0$ (inequalities involving matrices and vectors should be understood component-wise), and therefore an M-matrix (inverse monotone L-matrix). This can be proved using the following M-criterion, see [2] for instance.

Theorem 1. *Let A be an L-matrix and let there exist a vector w such that $w > 0$ and $Aw \geq \gamma$ for some positive constant γ. A is then an M-matrix and it holds that $\|A^{-1}\| \leq \gamma^{-1}\|w\|$.*

To see that A_N is an M-matrix, just set $w_i = 2 - x_i$, $i = 0, 1, \ldots, N$ in Theorem 1 to get that $A_N w \geq \min\{1, \beta\}$. This also implies that the discrete problem (5) is stable uniformly in ε,

$$\|A_N^{-1}\| \leq \frac{2}{\min\{1, \beta\}} \leq C. \tag{6}$$

Of course, the system (5) has a unique solution U^N.

4 A Bakhvalov-type mesh

A generalization of the Bakhvalov mesh [1] to a class of Bakhvalov-type meshes can be found in [9]. Here we take one of the Bakhvalov-type meshes from [9] for the discretization mesh I^N. We refer to this mesh as as Vulanović-Bakhvalov mesh (VB-mesh). The points of the VB-mesh are generated by the function λ in the sense that $x_i = \lambda(t_i)$, where $t_i = i/N$. The mesh-generating function λ is defined as follows:

$$\lambda(t) = \begin{cases} \psi(t), & t \in [0, \alpha], \\ \psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in [\alpha, 1], \end{cases} \tag{7}$$

with $0 < q < 1$ and $\psi = a\varepsilon\phi$, where

$$\phi(t) = \frac{t}{q - t} = \frac{q}{q - t} - 1, \quad t \in [0, \alpha].$$

On the interval $[\alpha, 1]$, λ is the tangent line from the point $(1, 1)$ to ψ, touching ψ at $(\alpha, \psi(\alpha))$. The point α can be determined from the equation

$$\psi(\alpha) + \psi'(\alpha)(1 - \alpha) = 1.$$

Since $\phi'(t) = q/(q - t)^2$, the above equation reduces to a quadratic one,

$$a\varepsilon\alpha(q - \alpha) + a\varepsilon q(1 - \alpha) = (q - \alpha)^2,$$

which is easy to solve for α:

$$\alpha = \frac{q - \sqrt{a\varepsilon q(1-q+a\varepsilon)}}{1+a\varepsilon}.$$

We have to assume that $a\varepsilon < q$ (which is equivalent to $\psi'(0) < 1$) and then $\alpha > 0$. Note also that $\alpha < q$ and

$$q - \alpha = \zeta\sqrt{\varepsilon}, \quad \zeta \le C, \quad \frac{1}{\zeta} \le C. \tag{8}$$

Let J be the index such that $t_{J-1} < \alpha \le t_J$. Starting from the mesh point x_J, the mesh is uniform, with step size H. However, x_J behaves differently from the transition point of the Shishkin mesh because

$$x_J \ge \psi(\alpha) = \frac{a\alpha}{\zeta}\sqrt{\varepsilon}.$$

We note that the transition point $\psi(\alpha)$ is different also from the Bakhvalov-Shishkin of Vulanović-Shishkin meshes in the sense of [7].

We now give the estimate for the condition number of A_N when the discrete problem (4) is formed on the VB-mesh as described above. The condition number is

$$\kappa(A_N) := \|A_N^{-1}\|\|A_N\|.$$

We estimate the upper bound for $\|A_N\|$ by examining the entries of the matrix A_N directly,

$$\|A_N\| \le C\frac{N^2}{\varepsilon}.$$

Combining this with (6), we get the following result.

Theorem 2. *The condition number of A_N on the VB-mesh satisfies the following sharp bound:*

$$\kappa(A_N) \le C\frac{N^2}{\varepsilon}.$$

5 Conditioning

Let $M = \text{diag}\,(m_0, m_1, \ldots, m_N)$ be a diagonal matrix with the entries

$$m_0 = 1, \quad m_i = \frac{\hbar_i}{H}, \ i = 1, 2, \ldots, N-1, \quad \text{and} \quad m_N = 1.$$

In other words,

$$m_0 = 1, \quad m_i = \frac{\hbar_i}{H}, \ i = 1, 2, \ldots, J, \quad \text{and} \quad m_i = 1, i = J+1, \ldots, N. \tag{9}$$

When the system (5) is multiplied by M, this is equivalent to multiplying the equations 1, 2, ..., J of the discrete problem (4) by $\hbar_i/H, i = 1, 2, \ldots, J$. The modified system is

$$\tilde{A}_N U^N = M\tilde{f}^N, \tag{10}$$

where $\tilde{A}_N = MA_N$. Let the entries of $\tilde{A}_N$ be denoted by $\tilde{a}_{ij}$, the nonzero ones being

$$l_i := \tilde{a}_{i-1,i} = \begin{cases} -\dfrac{\varepsilon}{h_i H}, & 1 \le i \le J-1, \\ -\dfrac{\varepsilon}{h_J H}, & i = J, \\ -\dfrac{\varepsilon}{H^2}, & J+1 \le i \le N-1, \end{cases}$$

$$r_i := \tilde{a}_{i,i+1} = \begin{cases} -\dfrac{\varepsilon}{h_{i+1} H} - \dfrac{b_i \hbar_i}{h_{i+1} H}, & 1 \le i \le J-1, \\ -\dfrac{\varepsilon}{H^2} - \dfrac{b_i \hbar_i}{H^2}, & i = J, \\ -\dfrac{\varepsilon}{H^2} - \dfrac{b_i}{H}, & J+1 \le i \le N-1, \end{cases}$$

and

$$d_i := \tilde{a}_{ii} = \begin{cases} 1, & i = 0 \\ -l_i - r_i + \dfrac{\hbar_i}{H} c_i, & 1 \le i \le J, \\ -l_i - r_i + c_i, & J+1 \le i \le N-1, \\ 1, & i = N. \end{cases}$$

Unlike the Shishkin mesh, which is piece-wise uniform, the VB-mesh is graded in the fine part. Because of this, it is more difficulty to prove the uniform stability of the modified scheme. This is done in Lemma 2 below, but first we need some crucial estimates for the graded mesh defined by (7).

Lemma 1. *For the mesh-generating function given in* (7)*, the following estimates hold true:*

$$\frac{\varepsilon(h_{i+1} - h_i)}{h_i h_{i+1}} \le \frac{2}{a}, \qquad i = 1, 2, \ldots, J-2, \tag{11}$$

and

$$\frac{\varepsilon(H - h_J)}{h_J H} \le \frac{\zeta\sqrt{\varepsilon}}{aq}. \tag{12}$$

Proof. For $i \le J-2$, we have

$$h_i = x_i - x_{i-1} = a\varepsilon \left(\frac{q}{q - t_i} - \frac{q}{q - t_{i-1}} \right) = \frac{a\varepsilon q}{N(q - t_{i-1})(q - t_i)},$$
$$h_{i+1} = \frac{a\varepsilon q}{N(q - t_i)(q - t_{i+1})},$$

and

$$h_{i+1} - h_i = \frac{2a\varepsilon q}{N^2(q - t_{i-1})(q - t_i)(q - t_{i+1})}.$$

Then (11) follows because

$$\frac{\varepsilon(h_{i+1}-h_i)}{h_i h_{i+1}} = \frac{2(q-t_i)}{aq} = \frac{2}{a}\left(1-\frac{t_i}{q}\right) \leq \frac{2}{a}.$$

The proof of (12) is more complicated due to the presence of h_J. First, $h_J = \gamma_1 + \gamma_2$, where $\gamma_1 = x_\alpha - x_{J-1}$, $\gamma_2 = x_J - x_\alpha$, and $x_\alpha = \psi(\alpha)$. Since

$$\begin{aligned}
\gamma_2 &= \psi'(\alpha)(t_J - \alpha) \\
&= \frac{a\varepsilon q}{q-\alpha}\left(\frac{t_J-\alpha}{q-\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 &= a\varepsilon\left(\phi(\alpha) - \phi\left(t_{J-1}\right)\right) \\
&= a\varepsilon\left(\frac{\alpha}{q-\alpha} - \frac{t_{J-1}}{q-t_{J-1}}\right) \\
&= \frac{a\varepsilon q}{q-\alpha}\cdot\frac{\alpha - t_{J-1}}{q-t_{J-1}},
\end{aligned}$$

we have

$$\begin{aligned}
h_J &= \frac{a\varepsilon q}{q-\alpha}\left[\frac{t_J-\alpha}{q-\alpha} + \frac{\alpha - t_{J-1}}{q-t_{J-1}}\right] \\
&= \frac{a\varepsilon q}{(q-\alpha)^2}\left[t_J - \alpha + \frac{(q-\alpha)(\alpha-t_{J-1})}{q-t_{J-1}}\right] \\
&= \frac{a\varepsilon q}{\zeta^2}\left[t_J - \alpha + \frac{\zeta\sqrt{\varepsilon}(\alpha-t_{J-1})}{q-t_{J-1}}\right].
\end{aligned}$$

Moreover,

$$\psi'(\alpha) = \frac{a\varepsilon q}{(q-\alpha)^2} \quad \text{and } H = x_{J+1} - x_J = \frac{\psi'(\alpha)}{N},$$

implying that

$$H = \frac{a\varepsilon q}{N(q-\alpha)^2}.$$

Therefore,

$$\begin{aligned}
H - h_J &= \frac{a\varepsilon q}{q-\alpha}\left[\frac{1}{N(q-\alpha)} - \frac{t_J-\alpha}{q-\alpha} - \frac{\alpha-t_{J-1}}{q-t_{J-1}}\right] \\
&= \frac{a\varepsilon q}{q-\alpha}\left[\frac{\alpha-t_{J-1}}{q-\alpha} - \frac{\alpha-t_{J-1}}{q-t_{J-1}}\right] \\
&= \frac{a\varepsilon q}{q-\alpha}\left(\alpha - t_{J-1}\right)\left[\frac{1}{q-\alpha} - \frac{1}{q-t_{J-1}}\right] \\
&= \frac{a\varepsilon q}{q-\alpha}\left(\alpha - t_{J-1}\right)\frac{\alpha - t_{J-1}}{(q-\alpha)(q-t_{J-1})} \\
&= \frac{a\varepsilon q}{(q-\alpha)^2}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}.
\end{aligned}$$

We now have

$$\begin{aligned}
\varepsilon\frac{H-h_J}{h_J H} &= \frac{a\varepsilon^2 q}{(q-\alpha)^2}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}\cdot\frac{q-\alpha}{a\varepsilon q}\cdot\frac{1}{\frac{t_J-\alpha}{q-\alpha}+\frac{\alpha-t_{J-1}}{q-t_{J-1}}}\cdot\frac{(q-\alpha)^2 N}{a\varepsilon q}\\
&= \frac{(q-\alpha)N}{aq}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}\cdot\frac{(q-\alpha)(q-t_{J-1})}{\frac{q}{N}-\alpha^2+2\alpha t_{J-1}-t_{J-1}t_J}\\
&= \frac{(q-\alpha)^2 N}{aq}\cdot\frac{(\alpha-t_{J-1})^2}{\omega} \le \frac{\zeta^2\varepsilon}{aqN}\cdot\frac{1}{\omega},
\end{aligned}$$

where

$$\omega := \frac{q}{N}-\alpha^2+2\alpha t_{J-1}-t_{J-1}t_J$$

and where in the last step we used (8) and the fact that $0 \le \alpha - t_{J-1} \le 1/N$. The denominator ω can be estimated as follows:

$$\begin{aligned}
\omega &= \frac{q}{N}-(\alpha-t_{J-1})^2-\frac{t_{J-1}}{N}\\
&= \frac{\zeta\sqrt{\varepsilon}+\alpha}{N}-(\alpha-t_{J-1})^2-\frac{t_{J-1}}{N}\\
&= \frac{\zeta\sqrt{\varepsilon}}{N}+\frac{1}{N}(\alpha-t_{J-1})-(\alpha-t_{J-1})^2\\
&= \frac{\zeta\sqrt{\varepsilon}}{N}+(\alpha-t_{J-1})(t_J-\alpha)\\
&\ge \frac{\zeta\sqrt{\varepsilon}}{N}, \quad \text{since } (\alpha-t_{J-1})(t_J-\alpha)\ge 0.
\end{aligned}$$

Therefore,

$$\varepsilon\frac{H-h_J}{h_J H} \le \frac{\zeta^2\varepsilon}{aqN}\cdot\frac{N}{\zeta\sqrt{\varepsilon}} = \frac{\zeta\sqrt{\varepsilon}}{aq}.$$

This completes the proof of (12). □

It is easy to see that $\tilde{A}_N$ is an L-matrix. The next lemma shows that $\tilde{A}_N$ is an M-matrix and that the modified discretization (10) is stable uniformly in ε.

Lemma 2. *Let ε be sufficiently small, independently of N, and let $a > 4/\beta$. Then the matrix $\tilde{A}_N$ of the system* (10) *satisfies*

$$\left\|\tilde{A}_N^{-1}\right\| \le C.$$

Proof. We want to construct a vector $v = [v_0, v_1, \ldots, v_N]^T$ such that

(a) $v_i \ge \delta$, $i = 0, 1, \ldots, N$, where δ is a positive constant independent of both ε and N,

(b) $v_i \le C$, $i = 0, 1, \ldots, N$,

(c) $\sigma_i := l_i v_{i-1} + d_i v_i + r_i v_{i+1} \ge \delta$, $i = 1, 2, \ldots, N-1$.

Then, according to the M-criterion,

$$\|\tilde{A}_N^{-1}\| \le \delta^{-1}\|v\| \le C.$$

The following choice of the vector v is motivated by [6, 11, 8]:

$$v_i = \begin{cases} \alpha - Hi + \lambda, & i \le J-1, \\ \alpha - Hi + \dfrac{\lambda}{1+\rho_J}(1+\rho)^{J-i}, & i \ge J, \end{cases}$$

where $\rho_J = \beta h_J/(2\varepsilon)$, $\rho = \beta H/(2\varepsilon)$, and α and λ are fixed positive constants. Since $HN \le C$, there exists a constant α such that $v_i \ge \alpha - Hi \ge \delta > 0$, so the condition (a) is satisfied. Then, because of $v_i \le \alpha + \lambda$, the condition (b) holds true if we show that $\lambda \le C$. We do this next as we verify the condition (c).

When $1 \le i \le J-2$, we use (11) to get

$$\begin{aligned} \sigma_i &= (l_i + d_i + r_i)v_i + l_i H - r_i H \\ &= \frac{\hbar_i}{H} c_i v_i - \frac{\varepsilon}{h_i} + \frac{\varepsilon}{h_{i+1}} + \frac{b_i \hbar_i}{h_{i+1}} \\ &\ge -\left(\frac{\varepsilon}{h_i} - \frac{\varepsilon}{h_{i+1}}\right) + \frac{b_i}{2} + \frac{b_i h_i}{2h_{i+1}} \\ &= -\frac{\varepsilon(h_{i+1} - h_i)}{h_i h_{i+1}} + \frac{b_i}{2} + \frac{b_i h_i}{2h_{i+1}} \\ &\ge -\frac{2}{a} + \frac{b_i}{2} \ge \frac{\beta}{2} - \frac{2}{a} =: \delta > 0. \end{aligned}$$

The constant δ exists because of the assumption $a > 4/\beta$.

For $i = J-1$, we have

$$\begin{aligned} \sigma_{J-1} &= \frac{\hbar_{J-1}}{H} c_{J-1} v_{J-1} + l_{J-1} H - r_{J-1} H \\ &\quad + \lambda l_{J-1} + \lambda d_{J-1} + r_{J-1} \frac{\lambda}{1+\rho_J} \\ &\ge -\frac{\varepsilon}{h_{J-1}} + \frac{\varepsilon}{h_J} + \frac{b_{J-1}\hbar_{J-1}}{h_J} - r_{J-1}\frac{\lambda\rho_J}{1+\rho_J} \\ &\ge -\frac{\varepsilon}{h_{J-1}} + \frac{b_{J-1}}{2} - r_{J-1}\frac{\lambda\rho_J}{1+\rho_J} \\ &\ge -\frac{\varepsilon}{h_{J-1}} + \frac{\beta}{2} + \left(\frac{\varepsilon}{h_J H} + \frac{b_{J-1}\hbar_{J-1}}{h_J H}\right)\frac{\lambda\beta h_J}{2\varepsilon + \beta h_J} \\ &= -\frac{\varepsilon}{h_{J-1}} + \frac{\beta}{2} + \left(\frac{2\varepsilon + b_{J-1}(h_{J-1} + h_J)}{2h_J H}\right)\frac{\lambda\beta h_J}{2\varepsilon + \beta h_J} \\ &\ge \frac{\beta}{2} - \frac{\varepsilon}{h_{J-1}} + \frac{\lambda\beta}{4H} \ge \frac{\beta}{2} > \delta \end{aligned}$$

with a suitable positive constant λ. We can choose such λ because the estimates $H \le 2N^{-1}$ and $q - t_{J-1} \le q - t_{J-2} \le 1$ imply

$$\frac{\lambda\beta}{4H} - \frac{\varepsilon}{h_{J-1}} = \frac{\lambda\beta}{4H} - \frac{N}{aq}(q - t_{J-1})(q - t_{J-2}) \ge N\left(\frac{\lambda\beta}{8} - \frac{1}{aq}\right) \ge 0.$$

For $i = J$, we get

$$\begin{aligned}
\sigma_J &= \frac{\hbar_J}{H} c_J v_J + l_J H - r_J H + \lambda \left[l_J + \frac{d_J}{1+\rho_J} + \frac{r_J}{(1+\rho_J)(1+\rho)} \right] \\
&\geq -\frac{\varepsilon}{h_J} + \frac{\varepsilon}{H} + \frac{b_J \hbar_J}{H} \\
&\quad + \frac{\lambda}{(1+\rho_J)(1+\rho)} \left[l_J(1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J \right] \\
&\geq \frac{\varepsilon}{H} - \frac{\varepsilon}{h_J} + \frac{b_J}{2} \\
&\quad + \frac{\lambda}{(1+\rho_J)(1+\rho)} \left[l_J(1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J \right] \\
&\geq \frac{\beta}{2} - \frac{\varepsilon(H-h_J)}{h_J H} \geq \delta > 0.
\end{aligned}$$

The above estimate holds true because (12) implies that

$$\frac{\varepsilon(H-h_J)}{h_J H} \leq \frac{\zeta\sqrt{\varepsilon}}{aq} \leq \frac{2}{a},$$

when ε is sufficiently small, and because we can show that

$$[l_J(1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J] \geq 0.$$

Indeed,

$$\begin{aligned}
l_J(1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J &= l_J\rho_J + l_J\rho_J\rho - r_J\rho \\
&= -\frac{\varepsilon}{h_J H}\frac{\beta h_J}{2\varepsilon} - \frac{\varepsilon}{h_J H}\frac{\beta h_J}{2\varepsilon}\frac{\beta H}{2\varepsilon} \\
&\quad + \left[\frac{\varepsilon}{H^2} + \frac{b_J \hbar_J}{H^2} \right] \frac{\beta H}{2\varepsilon} \\
&= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J \hbar_J}{2H\varepsilon} \\
&= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J (h_J + H)}{4H\varepsilon} \\
&\geq -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J}{4\varepsilon} \geq 0.
\end{aligned}$$

Finally, when $J+1 \leq i \leq N-1$, we have

$$\begin{aligned}
\sigma_i &= c_i v_i + l_i H - r_i H + \frac{l_i}{1+\rho_J} \left[\frac{\lambda}{(1+\rho)^{i-1-J}} - \frac{\lambda}{(1+\rho)^{i-J}} \right] \\
&\quad + \frac{r_i}{1+\rho_J} \left[\frac{\lambda}{(1+\rho)^{i+1-J}} - \frac{\lambda}{(1+\rho)^{i-J}} \right] \\
&\geq b_i + \frac{\rho(1+\rho)l_i - \rho r_i}{(1+\rho_J)(1+\rho)^{i+1-J}}\lambda \\
&\geq \frac{\beta}{2} + \frac{(l_i - r_i + l_i\rho)\rho}{(1+\rho_J)(1+\rho)^{i+1-J}}\lambda \\
&= \frac{\beta}{2} + \left(\frac{b_i}{H} - \frac{\beta}{2H} \right) \frac{\lambda\rho(1+\rho)^{J-i-1}}{1+\rho_J} \\
&\geq \frac{\beta}{2} > \delta.
\end{aligned}$$

$\square$

By examining the elements of the matrix $\tilde{A}_N$, we see that

$$\|\tilde{A}_N\| \leq CN^2.$$

When we combined this with Lemma 2, we get the following result.

Theorem 3. *The matrix $\tilde{A}_N$ of the system* (10) *satisfies*

$$\kappa(\tilde{A}_N) \leq CN^2.$$

6 Uniform convergence

Let τ_i, $i = 1, 2, \ldots, N-1$, be the consistency error of the finite-difference operator $\mathcal{L}^N$,

$$\tau_i = \mathcal{L}^N u_i - f_i.$$

We have

$$\tau_i = \tau_i[u] := \mathcal{L}^N u_i - (\mathcal{L}u)_i$$

and by Taylor's expansion we get that

$$|\tau_i[u]| \leq Ch_{i+1}(\varepsilon\|u'''\|_i + \|u''\|_i), \tag{13}$$

where $\|g\|_i := \max_{x_{i-1} \leq x \leq x_{i+1}} |g(x)|$ for any $C(I)$-function g. Let us define

$$\tilde{\tau}_i[u] = \begin{cases} \dfrac{\hbar_i}{H}\tau_i[u], & 1 \leq i \leq J, \\ \\ \tau_i[u], & J+1 \leq i \leq N-1. \end{cases}$$

Lemma 3. *The following estimate holds true for all $i = 1, 2, \ldots, N-1$:*

$$|\tilde{\tau}_i[u]| \leq CN^{-1}.$$

Proof. We use the decomposition (2) and estimates (3). For the smooth part of the solution, it is easy to show that $|\tilde{\tau}[s]| \leq CN^{-1}$. Then we need to show that

$$|\tilde{\tau}_i[y]| \leq CN^{-1}.$$

Case 1. Let $i \geq J+1$, i.e. $t_{i-1} \geq t_J \geq \alpha$. Then we have

$$\begin{aligned} |\tilde{\tau}_i[y]| = |\tau_i[y]| &\leq Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \\ &\leq CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(t_{i-1})/\varepsilon} \\ &\leq CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(\alpha)/\varepsilon} \\ &\leq CN^{-1}\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \\ &\leq CN^{-1}, \end{aligned}$$

where we have used the fact that $\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \leq C$.

Case 2. Let $i \le J$, i.e. $t_{i-1} < \alpha$, and at the same time, let $t_{i-1} \le q - 3/N$. Note that, when $t_{i-1} \le q - 3/N$, we have

$$t_{i+1} \le q - 1/N < q \quad \text{and} \quad q - t_{i+1} \ge \frac{1}{3}(q - t_{i-1}).$$

This is because

$$q - t_{i-1} \ge \frac{3}{N} \quad \Rightarrow \quad \frac{2}{3}(q - t_{i-1}) \ge \frac{2}{N},$$

which gives

$$q - t_{i+1} = q - t_{i-1} - \frac{2}{N} = \frac{1}{3}(q - t_{i-1}) + \frac{2}{3}(q - t_{i-1}) - \frac{2}{N} \ge \frac{1}{3}(q - t_{i-1}).$$

Therefore,

$$\begin{aligned}
|\tilde{\tau}_i[y]| = \frac{\hbar_i}{H}|\tau_i[y]| &\le \frac{\hbar_i}{H} C h_{i+1} \left(\varepsilon \|y'''\|_i + \|y''\|_i\right) \\
&\le CN^{-1} \left[\lambda'(t_{i+1})\right]^2 \varepsilon^{-2} e^{-\beta\lambda(t_{i-1})/\varepsilon} \\
&\le CN^{-1} \left[\phi'(t_{i+1})\right]^2 e^{-a\beta\phi(t_{i-1})} \\
&\le C\varepsilon^{-1} N^{-1} (q - t_{i+1})^{-4} e^{-a\beta(q/(q-t_{i-1})-1)} \\
&\le CN^{-1} (q - t_{i-1})^{-4} e^{-a\beta q/(q-t_{i-1})} \\
&\le CN^{-1},
\end{aligned}$$

because $(q - t_{i-1})^{-4} e^{-a\beta q/(q-t_{i-1})} \le C$.

Case 3. In the last case, we consider the remaining possibility, $q - 3/N < t_{i-1} < \alpha$. We use the fact that $\mathcal{L}y = 0$ to work with

$$|\tilde{\tau}_i[y]| = \frac{h_i}{H}|\tau_i[y]| \le \frac{\hbar_i}{H}\left(P_i + Q_i + R_i\right),$$

where

$$P_i = \varepsilon |D'' y_i|, \qquad Q_i = b_i |D' y_i|, \quad \text{and} \quad R_i = c_i |y_i|.$$

We now follow closely the technique in [10, Lemma 5], (see also [11, 8]), to get

$$\begin{aligned}
\frac{\hbar_i}{H}\left(P_i + Q_i + R_i\right) &\le C\left[\frac{\hbar_i}{H}\left(\frac{1}{\hbar_i}\varepsilon \cdot 2\|y'\|_i\right) + \frac{\hbar_i}{H}\left(\frac{1}{h_{i+1}}\|y\|_i\right) + e^{-\beta\lambda(t_i)/\varepsilon}\right] \\
&\le CNe^{-\beta\lambda(t_{i-1})/\varepsilon} \\
&\le CNe^{-a\beta\phi(t_{i-1})} \\
&\le CNe^{-a\beta\phi(q-3/N)} \\
&\le CNe^{-a\beta(qN/3-1)} \\
&\le CN^{-1}.
\end{aligned}$$

□

Remark 1. The technique used in the above proof is based on [9], where the same approach is successfully applied to reaction-diffusion problems. This approach is originally due to Bakhvalov [1]. The technique works here for convection-diffusion problems (1) because an extra ε-factor is obtained from the preconditioner (9).

When Lemmas 2 and 3 are combined, which amounts to the use of the consistency-stability principle, we obtain the following result.

Theorem 4. *Let ε be sufficiently small, independently of N, and let $a > 4/\beta$. Then the solution U^N of the discrete problem* (5) *on the VB-mesh satisfies*

$$\|U^N - u^N\| \leq CN^{-1},$$

where u is the solution of the continuous problem (1)*.*

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